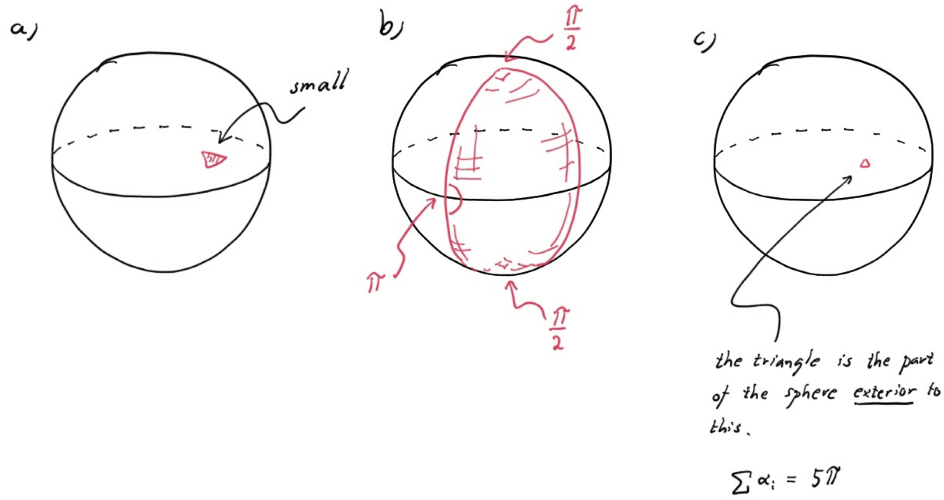
Answers to recommended problems

Chapter 2

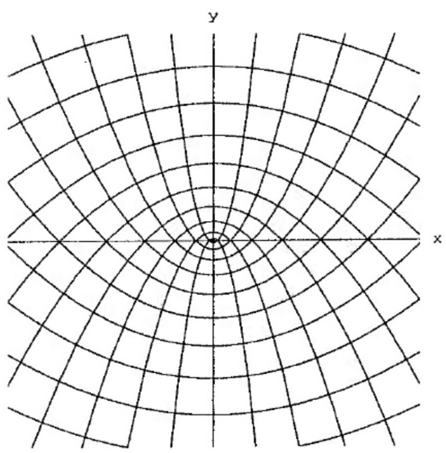
2:3 ---

2:4



2:5 $Area = 2\pi a^2(1 - \cos\frac{r}{a})$

2:7 (a)



(b) $dS^2 = (\mu^2 + \nu^2)(d\mu^2 + d\nu^2)$

(c) Yes, the curves intersect at right angles. (There is no cross term in the line-element.)

(d) $\mu^2 + \nu^2 = 2r$

(e) $C = 2\pi r$

Chapter 4

4:9 τ_{Ed} = 75 years, τ_{Joe} = 51 years, so difference is 24 years.

4:10 v = 0,36 c

4:13 4,24 m

4:15 Hint: Show first the following relation between the speed V in one of the frames and speed V' in the other frame (choose the relative speed v between the frames in the x-direction):

$$V'^2 - c^2 = (V^2 - c^2) \cdot \frac{(1 - v^2/c^2)}{(1 - v V^x / c^2)^2}$$

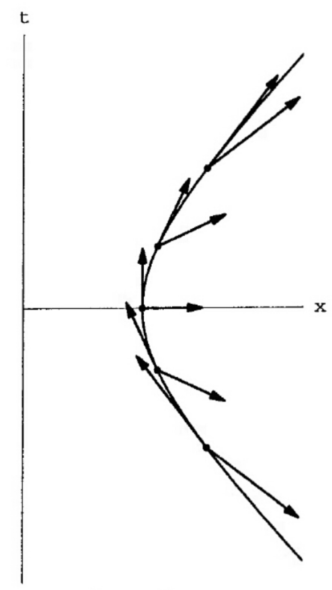
Since the factor to the right is positive the result follows.

Chapter 5

5:2 ---

5:4 $$a^\alpha = \gamma_v^4 (\vec{v} \cdot \vec{a} \,,\, \gamma_v^{-2} \vec{a} + (\vec{v} \cdot \vec{a}) \vec{v})$$

5:5



Yes, ***a*** and ***u*** are orthogonal.

5:7 $$s(\tau) = x_0 + \frac{c^2}{g} (\cosh \frac{g \tau}{c} - 1)$$

$$t(\tau) = \frac{c^2}{g} \sinh \frac{g \tau}{c}$$

5:8 (a) $u^\alpha = (\sqrt{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0)$ (in units where $c=1$)

(b) $p^\alpha = m u^\alpha$

5:11 $\Theta' = \Theta \pm \theta$

5:20 ---

Chapter 6

6:11 $M \sim 0{,}02\ M_{\text{sun}}$

6:12 Consider two points *a* and *b* on the surface of the sphere (which are not antipodal). There is one great circle through both points, defining two curves of extremal distance connecting the points, one shorter (segment 1) and one longer (segment 2). Segment 1 is also the path of shortest distance between *a* and *b*. But segment 2 is longer than any nearby path: Displacing all points of it in the same direction will make it shorter. Yet it is not the longest path: changing it into a zigzag line, or one winding around the sphere several times before reaching the other point, makes it longer.

6:13 $\tau_0 = T$ (observer standing on the ground)

$\tau_1 = T\left(1 + \frac{g^2 T^2}{32 c^2}\right)$ (observer carrying the clock att constant speed up and down)

$\tau_2 = T\left(1 + \frac{g^2 T^2}{24 c^2}\right)$ (observer throwing the clock)

Note that the longest proper time is registered by the clock that is thrown, since this is the “free fall path”.

If the clocks instead would have been carried in the horizontal direction, the longest time would have been registered by the clock at rest (as a consequence of just special relativity).

6:14 (a) $\Delta\tau_a = P\left(1 - \frac{3GM}{2Rc^2}\right)$

(b) $\Delta\tau_b \approx P\left(1 - \frac{GM}{Rc^2}\right)$ Note that $\Delta\tau_b > \Delta\tau_a$

(c) $\Delta\tau_c = 0$ So $\Delta\tau_b > \Delta\tau_a > \Delta\tau_c$

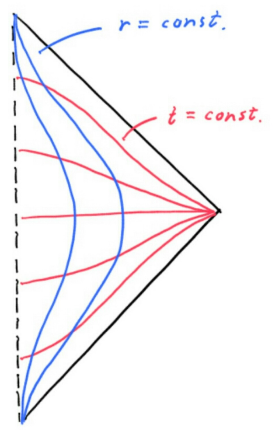
The worldline of the orbit is therefore neither the longest nor the shortest.

(d) If the particle is thrown radially outwards with the right velocity so that it returns in coordinate time P, that gives another curve of extremal proper time.

Chapter 7

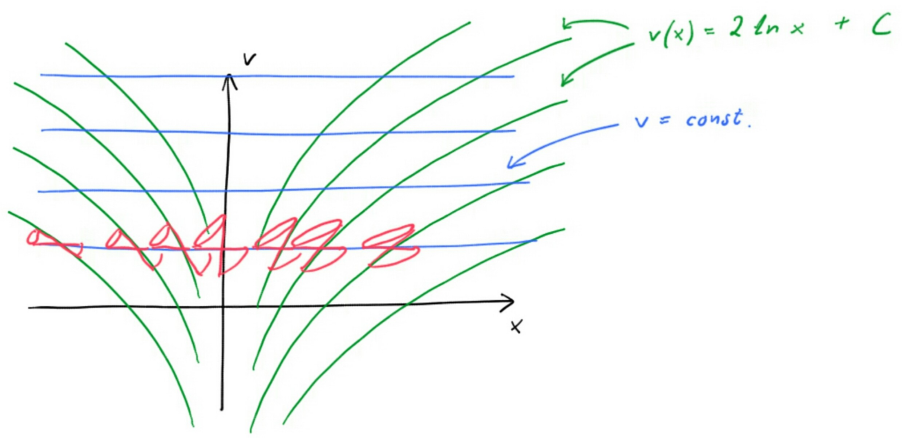
7:2 $t' = t - x$

7:4



7:5 (a) The two light rays through the point (v, x) is described by $dv = 0$ and $\dfrac{dv}{dx} = \dfrac{2}{x}$

(b) The null lines are described by $v = \text{const.}$ and $v(x) = 2 \ln x + C$



(c) The worldlines of particles must lie inside the light cone.

7:6 $$ds^2 = \frac{1}{\cos^2(t'-r')\ \cos^2(t'+r')}\left(-dt'^2 + dr'^2 + \sin^2 r' \cos^2 r' (d\theta^2 + \sin^2\theta\, d\phi^2)\right)$$

7:9 The numbers that should replace the question marks in the table of the problem are

40 40

100 80

This means that there are 100 – 80 = 20 second derivatives of the metric that cannot be transformed away by some choice of coordinates. These are the local measure of spacetime curvature.

7:11 [Hint: write down the 4-velocity along the spaceship trajectory and show that it is timelike.]

7:12 $\tau = T$

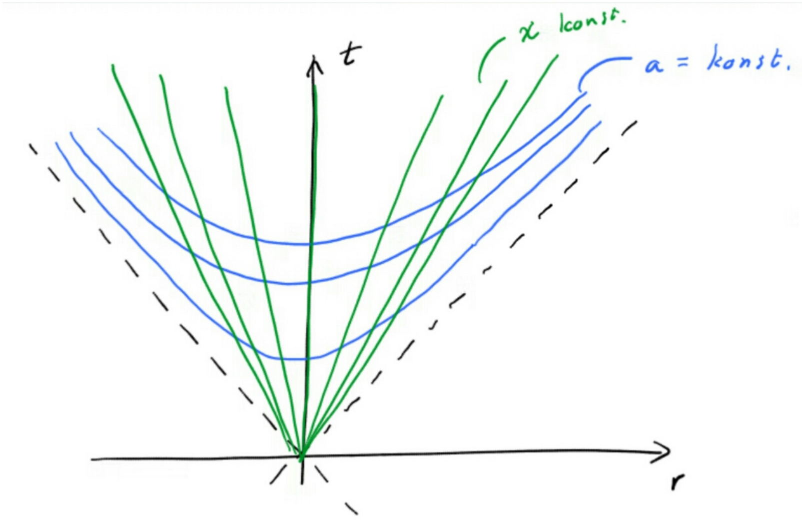
7:17 $V = 8\pi\left(b^2 R + \dfrac{R^3}{3}\right)$

7:18 (a) 3,05 M

(b) 215 M^3

7:20 $z(\rho) = 4M\sqrt{\dfrac{\rho}{2M} - 1}$

7:26 $ds^2 = -da^2 + a^2 d\chi^2 + a^2 \sinh^2\chi \, d\Omega^2$



Chapter 8

8:3 (a) $L = \left(\left(1-\dfrac{2M}{r}\right)\left(\dfrac{dt}{d\sigma}\right)^2 - \left(1-\dfrac{2M}{r}\right)^{-1}\left(\dfrac{dr}{d\sigma}\right)^2 - r^2\left(\dfrac{d\varphi}{d\sigma}\right)^2 \right)^{1/2}$

(b) $\dfrac{d^2 t}{d\tau^2} = -\left(1-\dfrac{2M}{r}\right)^{-1} \dfrac{2M}{r^2} \dfrac{dr}{d\tau} \dfrac{dt}{d\tau}$

$$\frac{d^2 r}{d\tau^2} = -\left(1-\frac{2M}{r}\right)\left[\frac{M}{r^2}\left(\frac{dt}{d\tau}\right)^2 - r\left(\frac{d\varphi}{d\tau}\right)^2 - \frac{M}{(r-2M)^2}\left(\frac{dr}{d\tau}\right)^2\right]$$

$$\frac{d^2\phi}{d\tau^2} = -\frac{2}{r}\frac{dr}{d\tau}\frac{d\phi}{d\tau}$$

(c) $$\Gamma^t{}_{tr} = \left(1-\frac{2M}{r}\right)^{-1}\frac{M}{r^2}$$

$$\Gamma^r{}_{tt} = \left(1-\frac{2M}{r}\right)\frac{M}{r^2}$$

$$\Gamma^r{}_{rr} = -\left(1-\frac{2M}{r}\right)^{-1}\frac{M}{r^2}$$

$$\Gamma^r{}_{\phi\phi} = -(r-2M)$$

$$\Gamma^\phi{}_{\phi r} = \frac{1}{r}$$

8:4 (a) ---

(b) $$-\frac{d^2x}{d\tau^2} - 2\Omega\frac{dy}{d\tau}\frac{dt}{d\tau} + \Omega^2 x\left(\frac{dt}{d\tau}\right)^2 = 0$$

$$-\frac{d^2y}{d\tau^2} + 2\Omega\frac{dx}{d\tau}\frac{dt}{d\tau} + \Omega^2 y\left(\frac{dt}{d\tau}\right)^2 = 0$$

$$\frac{d^2z}{d\tau^2} = 0$$

(c) In the non-relativistic limit $t = \tau$ and $\frac{dt}{d\tau} = 1$.
Hence, for example the *x*-equation becomes

$$\frac{d^2x}{dt^2} = -2\Omega\frac{dy}{dt} + \Omega^2 x$$

The first term on the right hand side is the Coriolis-force and the second the Centrifugal-force.

8:5 (See Hartle equation 8.18)

8:6 ---

8:8 $(-z, 0, x)$ and $(0, -z, y)$ (overall sign is arbitrary)

8:12 (a) ---

(b) $$\frac{d^2 x}{d S^2} = \frac{2}{y}\frac{dx}{dS}\frac{dy}{dS}$$

$$\frac{d^2 y}{d S^2} = -\frac{1}{y}\left(\frac{dx}{dS}\right)^2 + \frac{1}{y}\left(\frac{dy}{dS}\right)^2$$

Chapter 9

9:1 Inner and outer coordinate radius is 6*M* and 10 *M*, respectively. True distance between the shells are 4.46 *M*.

9:5 If ε were exactly equal to the maximum value of the potential, the particle coming in from infinity would spiral closer and closer to the unstable circular orbit at the potential maximum (but never quite reach it). If ε were slightly smaller than this, the particle would spiral for some time and then return to infinity. If ε were slightly larger the particle, after some turns, would end up in the black hole.

9:6 $$\tau = \frac{2M}{3}(3\sqrt{12} - 2) \approx 5{,}59\,M$$

9:7 $$\frac{v_{e=2}}{v_{e=1}} = \frac{\sqrt{10}}{2}$$

9:8 (a) $T \approx 116\,M$

(b) $T \approx 88\,M$

9:9 $$\frac{d\phi}{d\tau} = \left(\frac{M}{R^3}\right)^{1/2}\left(1 - \frac{3M}{R}\right)^{-1/2}$$

9:10 $$v = \left(\frac{M}{R}\right)^{1/2}\left(1 - \frac{2M}{R}\right)^{-1/2}$$

$v_{\mathrm{ISCO}} = 1/2$

9:12 $$v = \sqrt{\frac{2M/R}{1 - R^2/b^2}}$$

9:16 $27\pi M^2$

Chapter 12

12:3 ---

12:5 $\tau = 10\sqrt{5} M \int_0^1 \left(\frac{1}{s} - 1\right)^{-1/2} ds = 5\sqrt{5}\pi M$

12:8 Yes, an observer inside a black hole can receive information from the outside. But she will not be able to see all points outside, only the ones that are in her backward lightcone. (Hint: consider the observer in a Penrose diagram!)

12:9 The light rays that he fires will not cross his trajectory (as is clear from the Penrose diagram).

12:13 (a) An observer who falls into a black hole, feet first, will continue to see her own feet until her head hits the singularity. As she passes the horizon, she sees her feet at the same radius. As her head hits the singularity she sees her feet at some moments before they hit the singularity, so she never sees her own feet hit the singularity.

(b) It is not necessarily dark inside the black hole. Light from the collapsing star could be visible, as well as light from the outside.

12:14 $\tau = 2M \int_0^1 \sqrt{\frac{x}{1-x}}\, dx = \pi M$

12:15 $\frac{m_{esc}}{m} = \frac{(1 - 2M/R)^{1/2}}{1 + (2M/R)^{1/2}}$

This vanishes as $R = 2M$.

12:17 ---

12:21 ---

12:25 An observer in a Kruskal universe could not travel to “the other side”, nor see any light from stars in the other side. This is clear from the Penrose diagram on page 274, since light rays in this diagram is 45 degrees, and timelike lines is steeper than that. Hence, no timelike or lightlike line connects the interior of region II with the interior of region I.

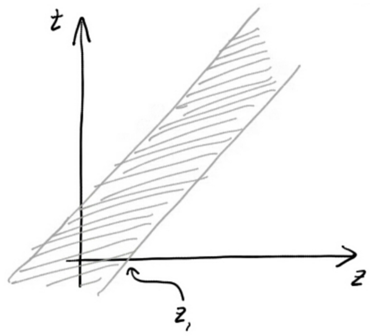
12:26 ---

12:27 ---

Chapter 16

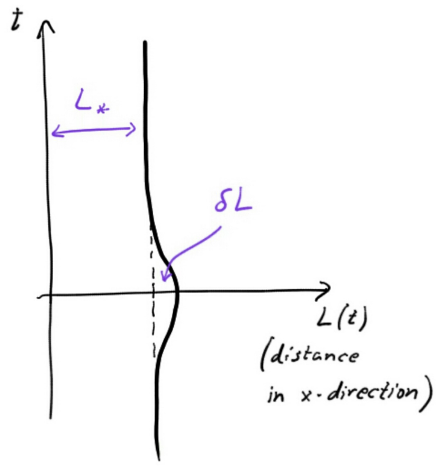
16:1 ---

16:2 (a)



$z_1 = \sigma\sqrt{\ln 2}$

(b)



$\text{Max}\,\dfrac{\delta L}{L_*} = \dfrac{a}{2}$

16:5 No risk!

16:8 (a) The ring would maintain its circular shape while oscillating in the x-direction.
(b) No, a similar motion pattern could not be reproduced with a gravitational wave.

Chapter 18

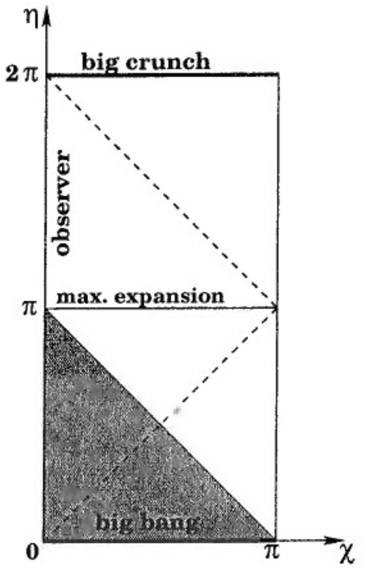
18:5 $$z = \frac{a(t_0)}{a(t_e)} - 1 = \frac{T(t_e)}{T(t_0)} - 1 \approx 1000$$

18:6 0.96 month

18:7 ---

18:11 (a) ---

(b)



(c) Already at maximum expansion, the observer can receive information from anywhere in the shaded region above, that is, from any *spatial* position. But, of course, all *spacetime* points cannot be seen before the observer hits the Big Crunch singularity.

(d) A light ray can just make it one turn around the universe from Big Bang to Big Crunch. So an observer cannot make that round trip. (Remember that $\chi = 0$ and $\chi = \pi$ are not the same, but opposite points on the spherical 3-space.)

18:17 ---

18:19 $$a(t) = \frac{1}{H}\cosh(Ht)$$

There is no initial singularity. The universe contracts, reaches its minimum size at $t = 0$ and then expands again.

18:21 ---

18:22 ---

18:23 ---

18:24 (a) $\rho_m = 2\rho_v = \frac{\Lambda}{4\pi}$

(b) $V = 2\pi^2 \Lambda^{-3/2}$

(c) The static universe is unstable.

18:29 ---

Chapter 20

20:3
$$a^x = \sin\theta\cos\phi\, a^r + r\cos\theta\cos\phi\, a^\theta - r\sin\theta\sin\phi\, a^\phi$$
$$a^y = \sin\theta\sin\phi\, a^r + r\cos\theta\sin\phi\, a^\theta + r\sin\theta\cos\phi\, a^\phi$$
$$a^z = \cos\theta\, a^r - r\sin\theta\, a^\theta$$
$$a_x = \sin\theta\cos\phi\, a_r + r^{-1}\cos\theta\cos\phi\, a_\theta - r^{-1}\sin^{-1}\theta\sin\phi\, a_\phi$$
$$a_y = \sin\theta\sin\phi\, a_r + r^{-1}\cos\theta\sin\phi\, a_\theta + r^{-1}\sin^{-1}\theta\cos\phi\, a_\phi$$
$$a_z = \cos\theta\, a_r - r^{-1}\sin\theta\, a_\theta$$

20:4
$$\nabla^\alpha f = g^{\alpha\beta}\nabla_\alpha f = \frac{1}{(2M)^2}\left(-\left(1-\frac{2M}{r}\right)^{-1} 10t \ , \ -\left(1-\frac{2M}{r}\right)4r \ , \ 0 \ , \ 0 \right)$$

20:5 (a) ---

(b)
$$(e_{\hat{\tau}})_\alpha = \left(-1 \ , \ -\left(\frac{2M}{R}\right)^{1/2}\left(1 - \frac{2M}{R}\right)^{-1} \ , \ 0 \ , \ 0 \right)$$
$$(e_{\hat{r}})_\alpha = \left(\left(\frac{2M}{R}\right)^{1/2} \ , \ \left(1 - \frac{2M}{R}\right)^{-1} \ , \ 0 \ , \ 0 \right)$$
$$(e_{\hat{\theta}})_\alpha = (0 \ , \ 0 \ , \ r \ , \ 0)$$
$$(e_{\hat{\phi}})_\alpha = (0 \ , \ 0 \ , \ 0 \ , \ r\sin\theta)$$

(c) $\quad (e^{\hat{\tau}})^{\alpha} = -(e_{\hat{\tau}})^{\alpha} \ , \ (e^{\hat{r}})^{\alpha} = (e_{\hat{r}})^{\alpha} \ , \ (e^{\hat{\theta}})^{\alpha} = (e_{\hat{\theta}})^{\alpha} \ , \ (e^{\hat{\phi}})^{\alpha} = (e_{\hat{\phi}})^{\alpha}$

(d) $\quad a_{\hat{\alpha}} = \boldsymbol{e}_{\hat{\alpha}} \cdot \boldsymbol{a} = (e_{\hat{\alpha}})_{\alpha} \cdot a^{\alpha} = \left(-3\left(\frac{4}{3}+3\left(\frac{2}{3}\right)^{1/2}\right) \ , \ 3\left(\frac{4}{3}\left(\frac{2}{3}\right)^{1/2}+3\right) \ , \ 0 \ , \ 0 \right)$

20:7 ---

20:10 $\quad \Gamma^{\alpha}_{\beta\gamma} = \frac{\partial x'^{\alpha}}{\partial x^{\delta}}\left(\frac{\partial}{\partial x'^{\beta}}\frac{\partial x^{\delta}}{\partial x'^{\gamma}}\right)$ which is symmetric in β and γ.

20:14 ---

20:15 ---

20:16 $\quad \kappa = \frac{-1}{4M}$ (with Hartle´s sign convention for κ)

20:17 ---

20:18 ---

20:20 ---

20:26 $\quad v = -\sqrt{\frac{2M}{r}}$

Chapter 21

21:4 ---

21:5 ---

21:6 ---

21:7 (a) ---

(b) (In n dimensions the Riemann tensor has $n^2(n^2-1)/12$ independent components, that is, 1 component in 2 dim., 6 components in 3 dim. and 20 components in 4 dim.)

21:11 The only non-vanishing components of the Riemann tensor is

$$R^1_{212}=R^2_{121}=\frac{-1}{y^2}$$

This gives that the only non-vanishing components of the Ricci tensor is

$$R_{11}=R_{22}=\frac{-1}{y^2}$$

Then

$$R=g^{\alpha\beta}R_{\alpha\beta}=-2$$

21:12 (a) $$R_{\hat{r}\hat{\theta}\hat{r}\hat{\theta}} = R_{\hat{r}\hat{\varphi}\hat{r}\hat{\varphi}} = \frac{-b^2}{(b^2+r^2)^2}$$

$$R_{\hat{\theta}\hat{\varphi}\hat{\theta}\hat{\varphi}}=\frac{b^2}{(b^2+r^2)^2}$$

(b) (The result follows from the fact that the Riemann tensor has no time components.)

(c) Consider the φ-component of the geodetic deviation equation:

$$\frac{d^2\chi^{\hat{\varphi}}}{d\tau^2} = \frac{b^2}{(b^2+r^2)^2}\chi^{\hat{\varphi}}v^2\gamma_v^2 \sim v^2 \quad \text{for non-relativistic speeds.}$$

(d) The tidal forces are largest at $r = 0$:

$$\frac{d^2\chi^{\hat{\varphi}}}{d\tau^2} = \frac{v^2}{b^2}\chi^{\hat{\varphi}}\gamma_v^2$$

For a comfortable trip b should be large and v should be small.

21:13 ---

21:14 ---

21:18 ---

21:21 ---

21:22 $$g^{\alpha\beta}=\eta^{\alpha\beta}-h^{\alpha\beta}$$

21:23 Let $f''(u)$ denote the second derivative of f with respect to its argument. Then

$$\delta R_{txtx} = -\frac{1}{2} f''(t-z)$$
$$\delta R_{zxzx} = -\frac{1}{2} f''(t-z)$$
$$\delta R_{tyty} = +\frac{1}{2} f''(t-z)$$
$$\delta R_{zyzy} = +\frac{1}{2} f''(t-z)$$
$$\delta R_{txzx} = +\frac{1}{2} f''(t-z)$$
$$\delta R_{tyzy} = -\frac{1}{2} f''(t-z)$$

21:24 Let X^{α} be the initial displacement of one of the test particles relative to some origin and χ^{α} the displacement vector as the wave passes. Then

$$\chi^x(t) = X^x\left(1 + \frac{f(t)}{2}\right)$$
$$\chi^y(t) = X^y\left(1 - \frac{f(t)}{2}\right)$$
$$\chi^z(t) = 0$$

Chapter 22

22:4 $(\text{torque}) = L^3(T^{xy} - T^{yx})$

Note that the moment of inertia is proportional to L^5, which then means that the angular acceleration is proportional to L^{-2}.

22:8 ---

22:9 ---

22:10 (a) $\mathrm{A} > 0$ and $\mathrm{A} > -\mathrm{B}$, $\mathrm{A} > -\mathrm{C}$, $\mathrm{A} > -\mathrm{D}$

(b) No.

22:12 (a) $f = A\,B\,C\,D$

(b) ---

22:13 ---

22:15 $T^{\hat{t}\hat{t}} = \frac{-1}{16\pi G} \cdot \frac{b^2}{(b^2 + r^2)^2} < 0$