Examination in General Relativity

May 31, 2018, 8:00 – 13:00

Please make sure that your answers are easy to read and have enough details to be followed. Calculations or reasoning should always be provided, unless otherwise stated in the problem. You are free to use formulas and expressions from the course book, but if you do, be careful to state its equation number, and explain why it is relevant to the solution.

Allowed help: Pocket calculator and course book (Hartle: Gravity).

Maximum points: 24 p. A passing grade requires at least 12 p.

1. The surface gravity of a black hole is in general defined as the constant κ in the equation

   $$\xi^\alpha \nabla_\alpha \xi^\beta = \kappa\, \xi^\beta$$

   where ξ is the Killing field tangent to the null geodesics that generate the horizon, and where the equation is evaluated at the horizon. In this problem, consider the special case of a Schwarzschild black hole, where the relevant Killing field is

   $$\xi^\alpha = (1, 0, 0, 0)$$

   in ordinary Schwarzschild coordinates.

   (a) What is the Killing field ξ as expressed in Eddington-Finkelstein coordinates? (1 p)

   (b) Find the value of the surface gravity for a Schwarzschild black hole of mass M, by evaluating the defining equation above in Eddington-Finkelstein coordinates! (4 p)

   [Modelled on Hartle problem 20:16]

2. Derive the expression

   $$R_{\alpha\beta\gamma\delta} = \frac{1}{2}\left(\partial_\gamma \partial_\beta g_{\alpha\delta} - \partial_\gamma \partial_\alpha g_{\beta\delta} - \partial_\delta \partial_\beta g_{\alpha\gamma} + \partial_\delta \partial_\alpha g_{\beta\gamma}\right)$$

   for the Riemann tensor in a local inertial frame (LIF) at a point p, starting from the general expressions

   $$R^\alpha{}_{\beta\gamma\delta} = \partial_\gamma \Gamma^\alpha{}_{\beta\delta} - \partial_\delta \Gamma^\alpha{}_{\beta\gamma} + \Gamma^\alpha{}_{\gamma\epsilon} \Gamma^\epsilon{}_{\beta\delta} - \Gamma^\alpha{}_{\delta\epsilon} \Gamma^\epsilon{}_{\beta\gamma}$$

   and

   $$\Gamma^\alpha{}_{\beta\gamma} = \frac{g^{\alpha\delta}}{2}\left(\partial_\gamma g_{\delta\beta} + \partial_\beta g_{\delta\gamma} - \partial_\delta g_{\beta\gamma}\right)$$

   (3 p)

   [Modelled on Hartle problem 21:6]

3. Consider a perfect fluid stress-energy tensor, as given in the local inertial frame where it is diagonal:

$$T_{\alpha\beta}=\begin{pmatrix}\rho & 0 & 0 & 0\\ 0 & p & 0 & 0\\ 0 & 0 & p & 0\\ 0 & 0 & 0 & p\end{pmatrix}$$

(a) What condition must hold between ρ and p so that any observer will see positive energy density no matter how fast that observer is moving with respect to the given frame? (This is called the weak energy condition.) (3 p)

(b) Consider vacuum energy, that is a $T_{\alpha\beta}$ such as the one above but with $p = -\rho$. Show that all observers measures the same value of the vacuum energy density, no matter how they are moving. (2 p)

[Modelled on Hartle problems 22:9 and 22:10]

4.

(a) Show that for FRW models (closed, open or flat) with any combination of matter and radiation but no vacuum energy, the scale factor $a(t)$ has a negative second time derivative (that is, for such models the expansion speed slows down). (3 p)

(b) Show that this is not always the case if there is a non-zero vacuum energy. (2 p)

[Modelled on Hartle problem 18:23]

5.

(a) Once across the event horizon of a Schwarzschild black hole of mass M, what is the longest proper time an observer can spend before being destroyed in the singularity? You may have use for the following integral:

$$\int_0^1 \sqrt{\frac{x}{1-x}}\,dx=\frac{\pi}{2}$$

(5 p)

(b) What is the numerical answer in seconds for the black hole in the center of our galaxy, which has a mass of approximately 4 million solar masses $M_{\odot}$ (where $M_{\odot} \approx 2 \cdot 10^{30}$ kg)? (1 p)

[Modelled on Hartle problem 12:14]