Examination in General Relativity

June 5, 2019, 8:00 – 13:00

Please make sure that your answers are easy to read and have enough details to be followed. Calculations or reasoning should always be provided, unless otherwise stated in the problem. You are free to use formulas and expressions from the course book, but if you do, be careful to state its equation number, and explain why it is relevant to the solution.

Allowed help: Pocket calculator and course book (Hartle: Gravity).

Maximum points: 24 p. A passing grade requires at least 12 p.

1. The geometry outside the Earth is well approximated by the weak field line element

$$ds^2 = -\left(1-\frac{2GM}{c^2 r}\right)(c\,dt)^2 + \left(1+\frac{2GM}{c^2 r}\right)dr^2 + r^2 d\Omega^2$$

Consider a satelite orbiting the Earth at a circular orbit of radius R. Let P be the period of the orbit as measured in coordinate time t (that is, as measured by a stationary observer far away). The relation between P and the radius R is then the usual one given by Kepler's third law:

$$P^2 = \frac{4\pi^2}{GM} R^3$$

(a) Calculate to first order in $1/c^2$ the proper time of the satelite for one turn around the Earth. (3 p)

(b) Calculate to first order in $1/c^2$ the time that it takes the satelite to travel once around the Earth according to a stationary observer at radius R. (2 p)

(c) Compare the times in (a) and (b) with P. Explain why the ordering of these three times is to be expected. (1 p)

[Modelled on Hartle problem 6:14]

2.

(a) Prove that the covariant derivative on the metric vanishes, that is,

$$\nabla_\alpha g_{\beta\gamma} = 0$$

by explicitly writing the covariant derivative of the metric in terms of the Christoffel symbols, and the Christoffel symbols in terms of the metric. (2 p)

[Modelled on Hartle problem 20:17]

(b) In a local inertial frame, the covariant derivative is just the partial derivative:

$$\nabla_\alpha v^\beta = \partial_\alpha v^\beta \qquad \text{(LIF)}$$

Use the tensor transformation law to transform this expression to a general frame, and use the result to prove that the Christoffel symbols defined by

$$\nabla_\alpha v^\beta = \partial_\alpha v^\beta + \Gamma^\beta{}_{\alpha\mu} v^\mu$$

must be symmetric in their two lower indices. (3 p)

[Modelled on Hartle problem 20:10]

3. Two particles fall in radially from infinity in the Schwarzschild geometry:

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2 d\Omega^2$$

One of the particles starts with $e = 1$, and the other with $e = 2$ (where e is the energy per unit mass as measured at infinity). A stationary observer at $r = 6M$ measures the speed of each particle when they pass by. How much faster is the second particle moving at that point according to the observer? (5 p)

[Modelled on Hartle problem 9:7]

4. Consider a closed FRW model, with matter but no radiation or vacuum energy:

$$ds^2 = -dt^2 + a^2(t)\left(d\chi^2 + \sin^2\chi\, d\Omega^2\right)$$

where the scale factor $a(t)$ can be expressed via the new time parameter η as

$$a(\eta) = \frac{\Omega}{2H_0(\Omega-1)^{3/2}}(1-\cos\eta)$$
$$t(\eta) = \frac{\Omega}{2H_0(\Omega-1)^{3/2}}(\eta-\sin\eta)$$

(Hartle equation 18:69)

(a) Show that, if η is used as time coordinate in the line element, it takes the form

$$ds^2 = a^2(\eta)\left[-d\eta^2 + d\chi^2 + \sin^2\chi\, d\Omega^2\right]$$

(1 p)

(b) Consider the (η,χ)-plane. What are the two families of light rays in this plane, expressed in these coordinates? (1 p)

(c) Draw a (η,χ) spacetime diagram indicating the Big Bang and the Big Crunch singularities. (1 p)

(d) At what times η can an observer receive information from all parts of this universe? (1 p)

(e) Is it possible for an observer to traverse the entire circumference of this universe? (1 p)

[Modelled on Hartle problem 18:11]

5. $x^\alpha(\tau)$ is a timelike geodesic parametrized by its proper time τ. It obeys the geodetic equation

$$u^\alpha \nabla_\alpha u^\beta = 0$$

where

$$u^\alpha = \frac{dx^\alpha}{d\tau}$$

Prove that if the curve is instead parametrized by some other parameter $\lambda(\tau)$, then the tangent

$$w^\alpha = \frac{dx^\alpha}{d\lambda}$$

obeys

$$w^\alpha \nabla_\alpha w^\beta = \kappa w^\beta$$

and obtain an expression for κ in terms of the function $\lambda(\tau)$ describing the relation between the two parametrizations. (3 p)

[Modelled on Hartle problem 20:15]